

Metric-Like Formulation Of the Spin-Three Gravity In Three Dimensions

Zhi-Qiang Guo*

*Departamento de Física y Centro Científico Tecnológico de Valparaíso,
Universidad Técnica Federico Santa María, Casilla 110-V, Valparaíso, Chile*

We provide a metric-like formulation of the spin-3 gravity in three dimensions. It is shown that the Chern-Simons formulation of the spin-3 gravity can be reformulated as a Einstein-Cartan-Sciama-Kibble theory coupled with the higher-spin matter fields. A duality-like transformation is also identified from this metric-like formulation.

PACS numbers: 11.10.Kk, 11.15.Yc, 04.50.Kd, 04.60.Rt

Introduction.—In three dimensions (3D), the pure Einstein gravity does not have local degrees of freedom [1]. The Einstein-Hilbert action with the cosmological constant term can be recast as a $SL(2, R) \times SL(2, R)$ Chern-Simons (CS) theory [2, 3], which is a manifestly topological theory. Recently, it was suggested that the higher-spin gravity [4, 5] in 3D could also be expressed as a CS theory [6, 7] but with the larger gauge group $SL(N, R) \times SL(N, R)$. In contrast with its concise formulation in terms of the frame-like fields, which facilitates the analysis of asymptotical symmetries [6–8] and higher-spin black hole solutions [9], a metric-like formulation of the higher-spin gravity are helpful to illuminate its geometrical structure and make its higher-spin freedoms transparent. From the perspective of anti-de Sitter/conformal field theory correspondence, the metric-like formulation in 3D is also useful to understand the thermodynamical properties (such as entropy [10] and shear viscosity [11]) of its dual theory in two dimensions [12]. However, a metric-like formulation can not be derived straightforwardly. A perturbative study of the metric-like formulation of the spin-3 gravity has been pursued in [10]. Geometrical analysis based on the metric compatibility method [13] shows that a complete metric-like formulation not only depends on the spin-2 field and the spin-3 field, but also higher-spin fields with more space-time indices are required. In this paper, we propose that if we assume the connection of the conventional spin-2 gravity has a torsion, then the CS formulation of the spin-3 gravity can be recast as a Einstein-Cartan-Sciama-Kibble theory (ECSK) [14], in which the spin-3 field can be regarded as the higher-spin matter acting as the source of the torsion.

Metric-Like Formulation.—Similar to its spin-2 cousin in three dimensions, the spin-3 gravity in 3D could be described by the $SL(3, R) \times SL(3, R)$ Chern-Simons theory

$$S = S_{\text{CS}}[A] - S_{\text{CS}}[\bar{A}], \quad (1)$$

$$S_{\text{CS}}[A] = \frac{k}{4\pi} \int \text{tr} (A \wedge dA + \frac{2}{3} A \wedge A \wedge A),$$

where $k = \frac{1}{16G}$. A and \bar{A} can further be decomposed into the frame-like fields

$$A = \omega + \frac{1}{l}e, \quad \bar{A} = \omega - \frac{1}{l}e. \quad (2)$$

Then the CS action (1) has the Palatini formulation

$$S = \frac{k}{\pi} \int \text{tr} (e \wedge (d\omega + \omega \wedge \omega) + \frac{1}{3l^2} e \wedge e \wedge e), \quad (3)$$

where the second term is the generalized cosmological term. The variation of ω yields the torsion constraints

$$\mathcal{T} = de + \omega \wedge e + e \wedge \omega = 0, \quad (4)$$

and the variation of e yields the equations of motion

$$\mathcal{R} = d\omega + \omega \wedge \omega + \frac{1}{l^2} e \wedge e = 0. \quad (5)$$

If we can solve ω in terms of e and de through the torsion equation (4), then a second-order formulation of the Palatini action can be obtained. For the Einstein gravity, ω and e take values in the Lie algebra of $SL(2, R)$, Eq. (4) can be solved straightforwardly, and a pure metric-like formulation can be achieved. For the spin-3 gravity, the Lie algebra of ω and e is $SL(3, R)$. A perturbative solution of Eq. (4) has been given in [10]. A non-perturbative attempt has been made in [13], which shows it is difficult to achieve a pure metric-like formulation of the CS action (1). The work of [13] is based on the $SL(3, R)$ invariant metric variables $\varphi_{\alpha\beta} = \text{tr}(e_\alpha e_\beta)$ and $\varphi_{\alpha\beta\gamma} = \text{tr}(e_\alpha e_\beta e_\gamma)$. Alternatively, in this paper, we use the $SL(2, R)$ decomposition of the $SL(3, R)$ Lie algebra

$$[J_a, J_b] = \epsilon_{abc} J^c, \quad [J_a, Q_{bc}] = \epsilon_{ab}^d Q_{dc} + \epsilon_{ac}^d Q_{db}, \\ [Q_{ab}, Q_{cd}] = \lambda^2 (\eta_{ac} \epsilon_{bdc} + \eta_{bc} \epsilon_{adc}) J^m + (c \leftrightarrow d), \quad (6)$$

where the small Latin letters take the values 0, 1, 2, and the definitions of η_{ab} and ϵ_{abc} follow the conventions in [7]. λ is a dimensionless constant. The anti-commutators of J_a furnish the Lie algebra of the $SL(2, R)$ group. Q_{ab} is symmetrical about its indices and satisfies the traceless condition $Q_{ab} \eta^{ab} = 0$. They transform as the 5D symmetrical representation of $SL(2, R)$. Using this realization of the $SL(3, R)$ Lie algebra, ω and e are expressed as

$$\omega = \omega^a J_a + \omega^{bc} Q_{bc}, \quad e = e^a J_a + e^{bc} Q_{bc}. \quad (7)$$

Here ω^{bc} and e^{bc} are also symmetrical and traceless, that is, $\omega^{bc} \eta_{bc} = 0$ and $e^{bc} \eta_{bc} = 0$. We define the metric-like fields from the frame-like fields as

$$g_{\alpha\beta} = e_\alpha^a e_\beta^b \eta_{ab}, \quad h_{\alpha\beta\gamma} = e_\alpha^{ab} e_\beta^c e_\gamma^d \eta_{ac} \eta_{bd}. \quad (8)$$

$g_{\alpha\beta}$ is the conventional $SL(2, R)$ invariant metric. $h_{\mu\alpha\beta}$ is only symmetrical about α and β , which belongs to the class of the mixed-symmetrical field discussed in [15]. Using $g^{\alpha\beta}$ as the inverse of $g_{\alpha\beta}$, $h_{\mu\alpha\beta}$ satisfies the traceless condition $h_{\mu\alpha\beta}g^{\alpha\beta} = 0$. We also have

$$e_\alpha^a e_\beta^b e_\gamma^c \epsilon_{abc} = \varepsilon_{\alpha\beta\gamma}, \quad E_a^\alpha E_b^\beta E_c^\gamma \epsilon^{abc} = \varepsilon^{\alpha\beta\gamma}, \quad (9)$$

where g is the determinant of $g_{\alpha\beta}$, and E_a^α is the inverse of e_α^a , which satisfies $E_a^\alpha e_\alpha^b = \delta_b^a$ and $E_a^\alpha e_\beta^a = \delta_\beta^\alpha$. We have defined

$$\varepsilon_{\alpha\beta\gamma} = \sqrt{-g} \epsilon_{\alpha\beta\gamma}, \quad \varepsilon^{\alpha\beta\gamma} = \frac{1}{\sqrt{-g}} \epsilon^{\alpha\beta\gamma}, \quad (10)$$

which are covariant antisymmetrical tensors under the general 3D coordinate transformation. By means of the $SL(2, R)$ variables, the torsion equation can be rewritten as

$$(\partial_\mu e_\nu^a + \omega_\mu^b e_\nu^c \epsilon_{bc}^a) - (\partial_\nu e_\mu^a + \omega_\nu^b e_\mu^c \epsilon_{bc}^a) \quad (11a)$$

$$= 4\lambda^2 (\omega_{\nu d}^b e_\mu^{dc} \epsilon_{bc}^a - \omega_{\mu d}^b e_\nu^{dc} \epsilon_{bc}^a),$$

$$(\partial_\mu e_\nu^{bc} + \omega_\mu^a e_\nu^{dc} \epsilon_{ad}^b + \omega_\mu^a e_\nu^{db} \epsilon_{ad}^c) - (\mu \leftrightarrow \nu) \quad (11b)$$

$$= (\omega_\nu^{ab} e_\mu^{dc} \epsilon_{ad}^c + \omega_\nu^{ac} e_\mu^{db} \epsilon_{ad}^b) - (\mu \leftrightarrow \nu).$$

Eqs. (11a) and (11b) have clear interpretations in term of the $SL(2, R)$ variables. The left side of Eq. (11a) can be interpreted as the torsion of the $SL(2, R)$ frame-like fields e_ν^a . The left side of Eq. (11b) transforms as a symmetrical representation of the the $SL(2, R)$ group. These observations provide us with the hints that Eqs. (11a) and (11b) can be reformulated as equations of metric-like fields through the assumptions

$$\partial_\mu e_\nu^a + \omega_\mu^b e_\nu^c \epsilon_{bc}^a = \Gamma_{\mu\nu}^\rho e_\rho^a \quad (12)$$

and

$$\omega_\mu^{bc} = \Omega_\mu^{\rho\sigma} e_\rho^b e_\sigma^c, \quad (13)$$

where $\Omega_\mu^{\rho\sigma}$ is symmetrical about ρ and σ , and it also satisfies the traceless condition $\Omega_\mu^{\rho\sigma} g_{\rho\sigma} = 0$. From Eq. (12), we can obtain the $SL(2, R)$ connection ω_μ^a

$$\omega_\mu^a = \frac{1}{2} \epsilon^{ab} e_\sigma^c (\partial_\mu e_\sigma^c - \Gamma_{\mu\sigma}^\rho e_\rho^c), \quad (14)$$

and Eq. (12) also yields the metric compatibility condition

$$\partial_\mu g_{\alpha\beta} = \Gamma_{\mu\alpha}^\rho g_{\rho\beta} + \Gamma_{\mu\beta}^\rho g_{\rho\alpha}, \quad (15)$$

which requires the connection to be

$$\Gamma_{\alpha\beta}^\rho = \bar{\Gamma}_{\alpha\beta}^\rho - g^{\rho\sigma} (T_{\alpha\sigma}^\tau g_{\tau\beta} + T_{\beta\sigma}^\tau g_{\tau\alpha}) + T_{\alpha\beta}^\rho, \quad (16)$$

$$\bar{\Gamma}_{\alpha\beta}^\rho = \frac{1}{2} g^{\rho\sigma} (\partial_\alpha g_{\sigma\beta} + \partial_\beta g_{\sigma\alpha} - \partial_\sigma g_{\alpha\beta}), \quad (17)$$

where $T_{\alpha\beta}^\rho$ is the torsion tensor, which is antisymmetric about α and β . In terms of the variables in Eqs. (8)-(9), (13) and (14), the action (3) can be rewritten as

$$S = \frac{1}{16\pi G} \int d^3x \sqrt{-g} \mathcal{L}, \quad (18)$$

$$\mathcal{L} = \mathcal{L}_1 + 4\lambda^2 (\mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4).$$

In Eq. (18), \mathcal{L}_1 is

$$\mathcal{L}_1 = R - \frac{2}{l^2}, \quad (19)$$

where

$$R_{\rho\mu\nu}^\sigma = \partial_\mu \Gamma_{\nu\rho}^\sigma - \partial_\nu \Gamma_{\mu\rho}^\sigma + \Gamma_{\mu\tau}^\sigma \Gamma_{\nu\rho}^\tau - \Gamma_{\nu\tau}^\sigma \Gamma_{\mu\rho}^\tau \quad (20)$$

is the Riemann curvature, and $R = g^{\alpha\beta} R_{\alpha\beta}^\sigma$ is the Ricci scalar. \mathcal{L}_2 , \mathcal{L}_3 and \mathcal{L}_4 are given by

$$\mathcal{L}_2 = g_{\alpha\beta} (\Omega_\rho^{\alpha\sigma} \Omega_\sigma^{\beta\rho} - \Omega_\sigma^{\alpha\sigma} \Omega_\rho^{\beta\rho}) \quad (21a)$$

$$\mathcal{L}_3 = \frac{1}{l^2} g_{\alpha\beta} (h_\rho^{\alpha\sigma} h_\sigma^{\beta\rho} - h_\sigma^{\alpha\sigma} h_\rho^{\beta\rho}), \quad (21b)$$

$$\mathcal{L}_4 = \varepsilon^{\mu\nu\alpha} (\nabla_\mu \Omega_\nu^{\rho\sigma} + T_{\mu\nu}^\tau \Omega_\tau^{\rho\sigma}) h_{\alpha\rho\sigma}, \quad (21c)$$

where

$$\nabla_\mu \Omega_\nu^{\alpha\beta} = \partial_\mu \Omega_\nu^{\alpha\beta} - \Gamma_{\mu\nu}^\sigma \Omega_\sigma^{\alpha\beta} + \Gamma_{\mu\sigma}^\alpha \Omega_\nu^{\sigma\beta} + \Gamma_{\mu\sigma}^\beta \Omega_\nu^{\alpha\sigma} \quad (22)$$

is the covariant derivative associated with the connection $\Gamma_{\mu\nu}^\sigma$. $h_\mu^{\alpha\beta} = g^{\alpha\rho} g^{\beta\sigma} h_{\mu\rho\sigma}$, that is, we always lower and raise the indices through $g_{\alpha\beta}$ and its inverse $g^{\alpha\beta}$. From the above, we saw that \mathcal{L}_1 is the action of the conventional spin-2 gravity with the cosmological constant. \mathcal{L}_4 is a topologically likewise coupling term. The meaning of \mathcal{L}_2 would be clear if we know the expression of $\Omega_\mu^{\rho\sigma}$. Now the action (18) has a metric-like formulation, but it is a first-order action about $\Omega_\mu^{\alpha\beta}$ and $h_{\mu\alpha\beta}$. In order to obtain a second-order formulation, we need to solve the torsion constraints (11a) and (11b). The torsion constraint (11a) can be reformulated as

$$-T_{\alpha\beta}^\gamma = 2\lambda^2 g_{\tau\mu} (\Omega_\alpha^{\sigma\tau} h_{\beta\sigma\rho} - \Omega_\beta^{\sigma\tau} h_{\alpha\sigma\rho}) \varepsilon^{\mu\rho\gamma}, \quad (23)$$

and the torsion constraint (11b) can be reformulated as

$$-K_{\alpha\beta}^\gamma = (\Omega_\alpha^{\gamma\tau} g_{\tau\beta} - \Omega_\beta^{\rho\tau} g_{\tau\alpha} \delta_\beta^\gamma) + (\alpha \leftrightarrow \beta), \quad (24a)$$

$$K_{\alpha\beta}^\gamma = \varepsilon^{\rho\sigma\gamma} (\nabla_\rho h_{\sigma\alpha\beta} + T_{\rho\sigma}^\tau h_{\tau\alpha\beta}). \quad (24b)$$

Eqs. (23) and (24a) are derived from Eqs. (11a) and (11b) by multiplying the frame-like fields E_a^α or e_β^b . Alternatively, they can also be derived through variations of the action (18) regarding to $T_{\alpha\beta}^\gamma$ and $\Omega_\mu^{\rho\sigma}$ respectively. Eqs. (23) and (24a) are coupling equations about $T_{\alpha\beta}^\gamma$ and $\Omega_\mu^{\rho\sigma}$. Eq. (23) demonstrates that the torsion is determined by the higher-spin fields, which provides the action (18) with the interpretation as a Einstein-Cartan-Sciama-Kibble theory [14]. The solution of Eq. (24a) can express the connection $\Omega_\mu^{\rho\sigma}$ with $h_{\mu\alpha\beta}$ and its derivatives. A solution of Eq. (24a) is

$$\Omega_\mu^{\alpha\beta} = \frac{1}{2} (g^{\alpha\sigma} K_{\mu\sigma}^\beta + g^{\beta\sigma} K_{\mu\sigma}^\alpha - \frac{2}{3} K_{\mu\sigma}^\sigma g^{\alpha\beta}) \quad (25)$$

$$- \frac{1}{2} g^{\alpha\rho} g^{\beta\sigma} g_{\mu\tau} K_{\rho\sigma}^\tau.$$

Through this expression, $\Omega_\mu^{\alpha\beta}$ can be eliminated from Eqs. (21a) and (21c), and Eqs. (21a) and (21c) can be regarded as the kinetic terms of the spin-3 fields $h_{\mu\alpha\beta}$.

Eq. (21c) looks like a Fierz-Pauli type massive term of $h_{\alpha\mu\nu}$. However, because the background solution of the action (18) is the anti-de Sitter space-time. Eq. (21c) plays the role to ensure the 3D diffeomorphism invariance of the action, but does not mean that the spin-3 field $h_{\mu\alpha\beta}$ is massive [15, 16]. We can further attempt to solve the torsion constraints (23). In 3D, $T_{\alpha\beta}^\gamma$ is equivalent to a rank (2,0) tensor through the definition

$$T^{\alpha\beta} = -\varepsilon^{\beta\rho\sigma} T_{\rho\sigma}^\alpha, \quad T_{\alpha\beta}^\gamma = \frac{1}{2} T^{\gamma\rho} \varepsilon_{\rho\alpha\beta}. \quad (26)$$

Substituting $\Omega_\mu^{\alpha\beta}$ into Eq. (23), we can obtain an equation of $T^{\alpha\beta}$

$$T^{\alpha\beta} + 4\lambda^2 T^{\rho\sigma} M_{\rho\sigma}^{\alpha\beta} = 4\lambda^2 \bar{\Omega}_\theta^{\sigma\tau} g_{\tau\mu} h_{\nu\sigma\rho} \varepsilon^{\mu\rho\alpha} \varepsilon^{\theta\nu\beta}, \quad (27)$$

where $\bar{\Omega}_\mu^{\alpha\beta}$ is defined as $\Omega_\mu^{\alpha\beta}$ in (25) but with the connection $\Gamma_{\alpha\beta}^\tau$ replaced by the Levi-Civita connection $\bar{\Gamma}_{\alpha\beta}^\tau$. $M_{\rho\sigma}^{\alpha\beta}$ is a complicated algebraic function of $g_{\alpha\beta}$ and $h_{\rho\alpha\beta}$, which does not have a compact expression. To solve $T^{\alpha\beta}$, we need to know the inverse of $(\delta_\rho^\alpha \delta_\sigma^\beta + 4\lambda^2 M_{\rho\sigma}^{\alpha\beta})$, which is obtainable perturbatively or non-perturbatively in an algebraic way through the Cayley-Hamilton method. The first order approximation of $T^{\alpha\beta}$ is given by the right side of Eq. (27). In this paper, we keep the torsion constraint (23) intact in order that the action (18) has a concise formulation, then the action (18) is a ECSK theory coupled with the higher-spin fields $h_{\rho\alpha\beta}$.

Equations of motion.—In order to obtain a transparent Lagrangian for $h_{\rho\alpha\beta}$, firstly we rewrite the Lagrangian \mathcal{L}_4 as

$$\mathcal{L}_4 = \varepsilon^{\mu\nu\alpha} (\nabla_\mu h_{\nu\rho\sigma} + T_{\mu\nu}^\tau h_{\tau\rho\sigma}) \Omega_\alpha^{\rho\sigma} \quad (28)$$

$$+ \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} \varepsilon^{\mu\nu\alpha} \Omega_\nu^{\rho\sigma} h_{\alpha\rho\sigma}).$$

The second line of this equation is a total divergence term. Substituting the solution (25) of $\Omega_\alpha^{\rho\sigma}$ into Eqs. (21a) and (28), we obtain a new Lagrangian

$$\mathcal{L}_2 + \mathcal{L}_4 = -\frac{1}{4} (g^{\mu\alpha} g^{\nu\beta} - g^{\mu\beta} g^{\nu\alpha}) \hat{\nabla}_\mu h_\nu^{\rho\sigma} \hat{\nabla}_\alpha h_{\beta\rho\sigma} \quad (29)$$

$$- \frac{1}{2} g^{\tau\theta} \varepsilon^{\mu\nu\rho} \varepsilon^{\alpha\beta\sigma} \hat{\nabla}_\mu h_{\nu\sigma\tau} \hat{\nabla}_\alpha h_{\beta\rho\theta},$$

where we have use $\hat{\nabla}_\mu h_{\nu\rho\sigma} = \nabla_\mu h_{\nu\rho\sigma} + T_{\mu\nu}^\tau h_{\tau\rho\sigma}$ to achieve a compact expression, and the divergence term in Eq. (28) was omitted. This Lagrangian has the Maxwell-like formulation, which is quadratic about the field strength. The identity

$$\varepsilon^{\mu\nu\rho} \varepsilon^{\alpha\beta\sigma} = g^{\mu\alpha} g^{\nu\sigma} g^{\rho\beta} + (g^{\nu\alpha} g^{\mu\beta} - g^{\mu\alpha} g^{\nu\beta}) g^{\rho\sigma} \quad (30)$$

$$- g^{\nu\alpha} g^{\mu\sigma} g^{\rho\beta} + (g^{\mu\sigma} g^{\nu\beta} - g^{\nu\sigma} g^{\mu\beta}) g^{\rho\alpha}$$

can be further used to rewrite Eq. (30) into a conventional formulation. Now we discuss the equations of motion about $g_{\alpha\beta}$ and $h_{\rho\alpha\beta}$. Their equations of motion are given

by the zero curvature condition (5), which can be decomposed into two equations as the torsion constraints (11a) and (11b). Firstly, from Eq. (5), we can obtain

$$2\lambda^2 \mathcal{T}_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \frac{1}{l^2} g_{\mu\nu}, \quad (31a)$$

$$\mathcal{T}_{\mu\nu} = \mathcal{L}_2 g_{\mu\nu} + 2(\Omega_\tau^{\sigma\tau} \Omega_{\nu\sigma\mu} - \Omega_\nu^{\sigma\tau} \Omega_{\tau\sigma\mu}) \quad (31b)$$

$$+ \mathcal{L}_3 g_{\mu\nu} + \frac{2}{l^2} (h_\tau^{\sigma\tau} h_{\nu\sigma\mu} - h_\nu^{\sigma\tau} h_{\tau\sigma\mu}).$$

Here $R_{\mu\nu} = R_{\mu\nu}^\sigma$ is the Ricci tensor. Eq. (31a) is the equation of motion of the spin-2 field $g_{\mu\nu}$, which has the same formulation with the Einstein equation. Because the connection has a torsion, $R_{\mu\nu}$ is not symmetric about its indices [14]. $\mathcal{T}_{\mu\nu}$ is the energy-momentum tensor contributed by the higher spin fields, and it is also not symmetric. From Eq. (5), we can also obtain

$$-H_{\alpha\beta}^\gamma = \frac{1}{l^2} (h_\alpha^{\gamma\tau} g_{\tau\beta} - h_\beta^{\rho\tau} g_{\tau\alpha} \delta_\beta^\gamma) + (\alpha \leftrightarrow \beta), \quad (32a)$$

$$H_{\alpha\beta}^\gamma = \varepsilon^{\rho\sigma\gamma} (\nabla_\rho \Omega_{\sigma\alpha\beta} + T_{\rho\sigma}^\tau \Omega_{\tau\alpha\beta}). \quad (32b)$$

Substituting the solution (25) of $\Omega_\mu^{\alpha\beta}$ into (32b), we can obtain the equations of motion about $h_{\mu\alpha\beta}$, though they do not have a compact expression as the action (29). We saw that Eqs. (32a) and (32b) have the same structure as Eqs. (24a) and (24b). From Eq. (32a), we have

$$\frac{1}{l^2} h_{\mu\alpha\beta} = \frac{1}{2} (g_{\alpha\sigma} H_{\mu\beta}^\sigma + g_{\beta\sigma} H_{\mu\alpha}^\sigma - \frac{2}{3} g_{\alpha\beta} H_{\mu\sigma}^\sigma) \quad (33)$$

$$- \frac{1}{2} g_{\mu\tau} H_{\alpha\beta}^\tau.$$

which is an equivalent formulation of Eq. (32a), and it is similar to Eq. (25).

Duality-like Transformation.—We have noticed that the similarity between Eq. (33) and Eq. (25), which indicate a duality-like transformation between $\Omega_\mu^{\alpha\beta}$ and $h_\mu^{\alpha\beta}$. To make this transformation transparent, we rewrite the Lagrangian \mathcal{L}_4 as

$$\mathcal{L}_4 = \frac{1}{2} \varepsilon^{\mu\nu\alpha} (\nabla_\mu \Omega_\nu^{\rho\sigma} + T_{\mu\nu}^\tau \Omega_\tau^{\rho\sigma}) h_{\alpha\rho\sigma} \quad (34)$$

$$+ \frac{1}{2} \varepsilon^{\mu\nu\alpha} (\nabla_\mu h_{\nu\rho\sigma} + T_{\mu\nu}^\tau h_{\tau\rho\sigma}) \Omega_\alpha^{\rho\sigma}$$

$$+ \frac{1}{2} \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} \varepsilon^{\mu\nu\alpha} \Omega_\nu^{\rho\sigma} h_{\alpha\rho\sigma}).$$

If we do not consider the divergence term in Eq. (34), then the action (18) is invariant under the duality like-wise transformation

$$\tilde{\Omega}_\mu^{\rho\sigma} = \frac{1}{l} h_\mu^{\rho\sigma}, \quad \frac{1}{l} \tilde{h}_\mu^{\rho\sigma} = \Omega_\mu^{\rho\sigma}. \quad (35)$$

We can further define

$$\Omega_\mu^{\rho\sigma} = \frac{1}{\sqrt{2}} (U_\mu^{\rho\sigma} - V_\mu^{\rho\sigma}), \quad (36a)$$

$$\frac{1}{l} h_\mu^{\rho\sigma} = \frac{1}{\sqrt{2}} (U_\mu^{\rho\sigma} + V_\mu^{\rho\sigma}), \quad (36b)$$

then \mathcal{L}_2 and \mathcal{L}_3 can be rewritten as

$$\mathcal{L}_2 + \mathcal{L}_3 = g_{\alpha\beta}(U_\rho^{\alpha\sigma}U_\sigma^{\beta\rho} - U_\sigma^{\alpha\sigma}U_\rho^{\beta\rho}) + g_{\alpha\beta}(V_\rho^{\alpha\sigma}V_\sigma^{\beta\rho} - V_\sigma^{\alpha\sigma}V_\rho^{\beta\rho}), \quad (37)$$

and \mathcal{L}_4 can be rewritten as

$$\mathcal{L}_4 = \frac{l}{2}\varepsilon^{\mu\nu\alpha}(\nabla_\mu U_\nu^{\rho\sigma} + T_{\mu\nu}^\tau U_\tau^{\rho\sigma})U_{\alpha\rho\sigma} - \frac{l}{2}\varepsilon^{\mu\nu\alpha}(\nabla_\mu V_{\nu\rho\sigma} + T_{\mu\nu}^\tau V_{\tau\rho\sigma})V_\alpha^{\rho\sigma} \quad (38)$$

up to the divergence term in Eq. (34). The torsion constraint (23) can be rewritten as

$$-T_{\alpha\beta}^\gamma = 2\lambda^2 g_{\tau\mu}(U_\alpha^{\sigma\tau}U_{\beta\sigma\rho} - V_\beta^{\sigma\tau}V_{\alpha\sigma\rho})\varepsilon^{\mu\rho\gamma}. \quad (39)$$

From the above, we saw that the cross products of $U_\mu^{\alpha\beta}$ and $V_\mu^{\alpha\beta}$ are eliminated from Eqs. (37) and (38). So the action (18) can be interpreted as the spin-2 gravity interacting with two rank (2,1) tensor fields. However, $U_\mu^{\alpha\beta}$ and $V_\mu^{\alpha\beta}$ are not free fields, and their interactions is provided by the torsion constraint (39) through the covariant derivative. From Eqs. (24a) and (32a), we have

$$-\tilde{H}_{\alpha\beta}^\gamma = \frac{1}{l}(U_\alpha^{\gamma\tau}g_{\tau\beta} - U_\rho^{\rho\tau}g_{\tau\alpha}\delta_\beta^\gamma) + (\alpha \leftrightarrow \beta), \quad (40a)$$

$$\tilde{H}_{\alpha\beta}^\gamma = \varepsilon^{\rho\sigma\gamma}(\nabla_\rho U_{\sigma\alpha\beta} + T_{\rho\sigma}^\tau U_{\tau\alpha\beta}). \quad (40b)$$

If we omit the torsion constraint (39), then Eq. (40a) is linear about $U_\alpha^{\beta\gamma}$. $\tilde{H}_{\alpha\beta}^\gamma$ can be interpreted as the field which is dual to the Maxwell-like field strength

$$\mathcal{F}_{\mu\nu\alpha\beta} = \nabla_\mu U_{\nu\alpha\beta} - \nabla_\nu U_{\mu\alpha\beta}. \quad (41)$$

While the right side of Eq. (40a) is a part of $U_{\alpha\beta\gamma}$ which is symmetric and traceless about its first two indices. So Eq. (40a) has the meaning that the dual of the field strength of $U_{\alpha\beta\gamma}$ is the minus of its symmetric and traceless part about its first two indices. Similarly, from Eqs. (24a) and (32a), we also have

$$\tilde{K}_{\alpha\beta}^\gamma = \frac{1}{l}(V_\alpha^{\gamma\tau}g_{\tau\beta} - V_\rho^{\rho\tau}g_{\tau\alpha}\delta_\beta^\gamma) + (\alpha \leftrightarrow \beta), \quad (42a)$$

$$\tilde{K}_{\alpha\beta}^\gamma = \varepsilon^{\rho\sigma\gamma}(\nabla_\rho V_{\sigma\alpha\beta} + T_{\rho\sigma}^\tau V_{\tau\alpha\beta}), \quad (42b)$$

which have the interpretations similar to Eqs. (40a) and (40b).

Generalized diffeomorphism.—Now we discuss the potential symmetries of the action (18). These symmetries can be induced from the symmetries of the Chern-Simons action (1). If the boundary terms are negligible, then the CS action (1) is invariant under the infinitesimal $SL(3, R) \times SL(3, R)$ gauge transformations

$$\delta A = d\zeta + [A, \zeta], \quad (43a)$$

$$\delta \bar{A} = d\bar{\zeta} + [\bar{A}, \bar{\zeta}]. \quad (43b)$$

In terms of the decomposition (2), we have

$$\delta\omega = d\Lambda + [\omega, \Lambda] + \frac{1}{l}[e, \xi], \quad (44)$$

$$\frac{1}{l}\delta e = d\xi + [\omega, \xi] + \frac{1}{l}[e, \Lambda], \quad (45)$$

where we have defined

$$\xi = \frac{1}{2}(\zeta - \bar{\zeta}), \quad \Lambda = \frac{1}{2}(\zeta + \bar{\zeta}). \quad (46)$$

Now we focus on the transformation (45). Λ and ξ have the $SL(2, R)$ decomposition

$$\Lambda = \Lambda^a J_a + \Lambda^{bc} Q_{bc}, \quad \xi = \xi^a J_a + \xi^{bc} Q_{bc}, \quad (47)$$

where Λ^{ab} and ξ^{ab} are symmetrical and traceless. If $\xi = 0$, then Eq. (45) yields the local Lorentz transformations

$$\delta_\Lambda g_{\mu\nu} = 4\lambda^2 (h_\mu^{\rho\sigma} \Lambda_{\rho\tau} g^{\tau\theta} \varepsilon_{\sigma\theta\nu} + (\mu \leftrightarrow \nu)), \quad (48a)$$

$$\delta_\Lambda h_{\alpha\mu\nu} = (\Lambda_{\mu\rho} g^{\rho\theta} \varepsilon_{\alpha\theta\nu} + (\mu \leftrightarrow \nu)) + 4\lambda^2 (h_\alpha^{\rho\sigma} h_\mu^{\tau\theta} g_{\rho\nu} \Lambda_{\tau\beta} g^{\beta\gamma} \varepsilon_{\theta\sigma\gamma} + (\mu \leftrightarrow \nu)), \quad (48b)$$

where $\Lambda_{\mu\nu} = \Lambda_{ab} e_\mu^a e_\nu^b$ is symmetrical and satisfies the traceless condition $g^{\mu\nu} \Lambda_{\mu\nu} = 0$. We saw that the local Lorentz transformations only depend on the parameter $\Lambda_{\rho\sigma}$, but it is independent of $\Lambda_\mu = \Lambda_a e_\mu^a$. This is consistent with the fact that $g_{\mu\nu}$ and $h_{\alpha\mu\nu}$ are $SL(2, R)$ invariant variables, but they are not $SL(3, R)$ invariant ones. Otherwise, If $\Lambda = 0$, from Eq. (45), we can obtain the generalized diffeomorphism

$$\frac{1}{l}\delta_\xi g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu \quad (49a)$$

$$+ 4\lambda^2 (\Omega_\mu^{\tau\beta} \xi_{\tau\rho} g^{\rho\sigma} \varepsilon_{\beta\sigma\nu} + (\mu \leftrightarrow \nu)),$$

$$\frac{1}{l}\delta_\xi h_{\alpha\mu\nu} = \nabla_\alpha \xi_{\mu\nu} + (h_{\alpha\rho\mu} g^{\rho\gamma} \nabla_\nu \xi_\gamma + (\mu \leftrightarrow \nu)) + 4\lambda^2 (h_{\alpha\rho\mu} g^{\rho\gamma} \Omega_\nu^{\tau\beta} \xi_{\tau\theta} g^{\theta\sigma} \varepsilon_{\beta\sigma\gamma} + (\mu \leftrightarrow \nu)), \quad (49b)$$

where $\nabla_\mu \xi_\nu = \partial_\mu \xi_\nu - \Gamma_{\mu\nu}^\rho \xi_\rho$ is the covariant derivative with the connection (16). $\xi_\mu = \xi_a e_\mu^a$, and $\xi_{\mu\nu} = \xi_{ab} e_\mu^a e_\nu^b$ is symmetrical and traceless. $\Omega_\nu^{\tau\beta}$ is defined by Eq. (25). If $h_{\alpha\mu\nu}$ is small, then the second term of the right side of Eq. (49a) is negligible. Eq. (49a) yields the conventional diffeomorphism for the spin-2 gravity.

Conclusions.—We have provided a metric-like formulation for the $SL(3, R) \times SL(3, R)$ Chern-Simons theory using the $SL(2, R)$ invariant variables. This metric-like formulation can be interpreted as a Einstein-Cartan-Sciama-Kibble theory [14], in which the torsion is determined by the higher-spin fields. The local Lorentz transformation and the generalized diffeomorphism can be expressed with these metric-like fields manifestly. We also identify a duality-like transformation in this metric-like formulation. Because the Lie algebra of $SL(N, R)$ has the decomposition under its sub algebra $SL(2, R)$ similar to that of $SL(3, R)$, the $SL(2, R)$ variables used here could also be useful to find a metric-like formulation for the $SL(N, R) \times SL(N, R)$ Chern-Simons theory [7], and the duality-like transformation discussed here could also be found in those theories.

Acknowledgments.—This work was supported in part by Fondecyt (Chile) grant 1100287 and by Project Basal under Contract No. FB0821.

* zhiqiang.guo@usm.cl

- [1] S. Deser, R. Jackiw, and G. 't Hooft, *Annals Phys.* **152**, 220 (1984).
- [2] A. Achucarro and P. K. Townsend, *Phys. Lett. B* **180**, 89 (1986).
- [3] E. Witten, *Nucl. Phys. B* **311**, 46 (1988).
- [4] M. A. Vasiliev, *Phys. Lett. B* **243**, 378 (1990).
- [5] M. Blencowe, *Class. Quant. Grav.* **6**, 443 (1989).
- [6] M. Henneaux and S.-J. Rey, *JHEP* **1012**, 007 (2010), [arXiv:1008.4579 \[hep-th\]](#).
- [7] A. Campoleoni, S. Fredenhagen, S. Pfenninger, and S. Theisen, *JHEP* **1011**, 007 (2010), [arXiv:1008.4744 \[hep-th\]](#).
- [8] J. D. Brown and M. Henneaux, *Commun. Math. Phys.* **104**, 207 (1986).
- [9] M. Gutperle and P. Kraus, *JHEP* **1105**, 022 (2011), [arXiv:1103.4304 \[hep-th\]](#).
- [10] A. Campoleoni, S. Fredenhagen, S. Pfenninger, and S. Theisen, *J. Phys. A* **46**, 214017 (2013), [arXiv:1208.1851 \[hep-th\]](#).
- [11] G. Policastro, D. T. Son, and A. O. Starinets, *Phys. Rev. Lett.* **87**, 081601 (2001), [arXiv:hep-th/0104066 \[hep-th\]](#).
- [12] M. R. Gaberdiel and R. Gopakumar, *J. Phys. A* **46**, 214002 (2013), [arXiv:1207.6697 \[hep-th\]](#).
- [13] I. Fujisawa and R. Nakayama, *Class. Quant. Grav.* **30**, 035003 (2013), [arXiv:1209.0894 \[hep-th\]](#).
- [14] F. Hehl, P. Von Der Heyde, G. Kerlick, and J. M. Nester, *Rev. Mod. Phys.* **48**, 393 (1976).
- [15] A. Campoleoni and D. Francia, *JHEP* **1303**, 168 (2013), [arXiv:1206.5877 \[hep-th\]](#).
- [16] S. Deser and A. Waldron, *Phys. Rev. Lett.* **87**, 031601 (2001), [arXiv:hep-th/0102166 \[hep-th\]](#).